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Remarks on Linear Selections for the Metric Projection

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The aim of this paper is to give a characterization of the finite-dimensional subspaces of L_p, $1 \leqslant p < \infty$, and $C_0(T)$ whose metric projections admit linear selections. The paper also gives a characterization of finite co-dimensional subspaces of l_1 and c_0 whose metric projections have linear selections.

1. INTRODUCTION

A linear subspace M of a normed linear space X is called *proximinal* (resp. *Chebyshev*) if, for each x in X, the set of best approximations to x from M, i.e., the set

$$P_M(x) = \{ y \in M : \|x - y\| = \inf_{m \in M} \|x + m\| \}$$

is nonempty (resp. a singleton). The set-valued mapping $P_M : X \to 2^M$ thus defined is called the *metric projection* onto M. A *selection* for P_M is a function $s : X \to M$ such that $s(x) \in P_M(x)$ for every $x \in X$. Let M^0 denote

$$\{x \in X : \|x\| = \inf_{m \in M} \|x - m\|\}.$$

It is known [5] that P_M has a linear selection if and only if M^0 contains a closed subspace N such that $X = M + N$.

In Section 2, we study the linear metric projections on $L_p = L_p(T, \Sigma, \mu)$. Let A_0 denote a union of atoms in (T, Σ, μ) and let $A_1 = T - A_0$. For an n-dimensional subspace M of L_p, we prove the following theorem. P_M has a linear selection if and only if there exist k disjoint measurable subsets $B_1, B_2, \ldots, B_k$ of A_0 such that $M = \oplus_{i=1}^k M_i$ and M_i is either $L_p(B_i)$ or a hyperplane of $L_p(B_i)$, where $L_p(B_i)$ is the set

$$\{f \in L_p : \operatorname{supp}(f) \subseteq B_i\}.$$

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If M is a n-co-dimensional subspace of L_p, we prove the following theorem. P_M admits a linear selection if and only if there exist n disjoint measurable sets $T_1, T_2, \ldots, T_n$ such that $M = \oplus_{i=1}^n M_i \oplus L_p(T - \bigcup_{i=1}^n T_i)$, where M_i is a hyperplane of $L_p(T_i)$.

In Section 3, we consider the space $L_1 = L_1(T, \Sigma, \mu)$ of integrable functions on the measure space (T, Σ, μ). For an n-dimensional subspace M of L_1, we prove the following theorem. P_M has a linear selection if and only if there exists a subset T_0 of T which contains exactly n atoms such that for each $m \in M$

$$\int_{T_0} |m(t)| \, d\mu \geqslant \int_{T-T_0} |m(t)| \, d\mu.$$

Let (e_i) be the natural basis of l_1. For any subspace M of l_1, we prove the following theorem. P_M has a linear selection if and only if there exists a subset $S \subseteq N$ such that $\operatorname{span}\{e_i : i \in S\}$ is a complement of M and

$$\sum_{i \notin S} |m(i)| \geqslant \sum_{i \in S} |m(i)|$$

for each $m \in M$.

In Section 4, we consider the space $C_0(T)$ of all real-valued continuous functions on the locally Hausdorff space T which vanish at infinity. Let A_0 be the union of all isolated points of T. For an n-dimensional subspace M of $C_0(T)$, we prove that P_M admits a linear selection if and only if there exist k disjoint subsets $B_1, B_2, \ldots, B_k$ of A_0 such that $M = \oplus_{i=1}^k M_i$, where M_i is either $C(B_i)$ or a hyperplane of $C(B_i)$. For an n-co-dimensional subspace M of c_0, we prove that P_M has a linear selection if and only if there exist n disjoint finite subsets $B_1, B_2, \ldots, B_n$ of $\mathbb{N}$ such $M = \oplus_{i=1}^n M_i \oplus c_0(\mathbb{N} - \bigcup_{i=1}^n B_i)$, where M_i is a hyperplane of $C(B_i)$.

2. Linear Selections in L_p, $1 < p < \infty$

Let (T, Σ, μ) be a measure space, and let $L_p = L_p(T, \Sigma, \mu)$, $1 \leqslant p < \infty$, denote the space of all real-valued measurable functions x on T whose absolute pth powers are integrable and whose norm is

$$\|x\| = \left[\int_T |x(t)|^p \, dt\right]^{1/p}.$$

An *atom* is a set $A \in \Sigma$ such that $0 < \mu(A) < \infty$ and if B is a measurable subset of A then either $\mu(A) = \mu(B)$ or $\mu(B) = 0$. Hence, any measurable function x is constant a.e. (μ) on an atom A, and we can assume that every atom contains only one point. For $x \in L_p$, the *support* of x and *zero set* of x

are defined (up to a set of measure zero) by $\mathrm{supp}(x) = \{t \in T: x(t) \neq 0\}$ and $Z(x) = T - \mathrm{supp}\, x = \{t \in T: x(t) = 0\}$. If $x \in L_p$, we will denote by $[x]$ the one-dimensional subspace spanned by x.

Suppose M is a subspace of L_p, $1 \leqslant p < \infty$, such that P_M admits a linear metric selection s. Then $I - s$ is a contractive projection. The range N of such a projection ([1], also see Theorem 3 of [8, p. 162]) is of the form

$$N = \{fg: f \in L_p(T, \Sigma_0, \nu)\},$$

where Σ_0 is a subring of Σ, $g \in L_p(T, \Sigma, \mu)$ and $d\nu = |g|^{-p}\, d\mu$.

We need the following characterization of best approximations from subspaces of L_p, $1 < p < \infty$.

LEMMA 1. [4] *Let $0 \neq y \in L_p$, $1 < p < \infty$, and $x \in L_p$. Then $x \in [y]^0$ if and only if*

$$\int_T y \,\mathrm{sgn}\, x\, |x|^{p-1}\, d\mu = 0.$$

Hence, if $1 < p < \infty$, M is a subspace of

$$\bar{M} = \left\{ y \in L_p : \int_T y \,\mathrm{sgn}\, x\, |x|^{p-1}\, d\mu = 0 \text{ for all } x \in N \right\}$$

$$= \left\{ y \in L_p : \int_A y\, |g|^{p-1} \,\mathrm{sgn}\, g\, d\mu = 0 \text{ for all } A \in \Sigma_0 \right\},$$

and $M + N = L_p$. Clearly $\bar{M}$ is a complement of N, and $\bar{M} = M$. We have the following theorem.

THEOREM 2. *Let M be a subspace of L_p, $1 < p < \infty$. Then P_M admits a linear selection if and only if there exist $g \in L_p$ and a subring Σ_0 of Σ such that*

$$M = \{ y \in L_p : \int_A y \,\mathrm{sgn}\, g\, |g|^{p-1}\, d\mu = 0 \text{ for all } A \in \Sigma_0 \}.$$

Let A_0 denote a union of atoms in (T, Σ, μ) and let $A_1 = T - A_0$. We have the following corollary.

COROLLARY 3. *Suppose M is an n-dimensional subspace of L_p, $1 < p < \infty$. The following properties are equivalent.*

(i) *P_M admits a linear selection s.*

(ii) *There exist k disjoint subsets $B_1, B_2, \ldots, B_k$ of A_0 such that $M = \oplus_{i=1}^k M_i$, where M_i is either $L_p(B_i)$ or a hyperplane of $L_p(B_i)$.*

Proof. (i) $\Rightarrow$ (ii). The range N of $I-s$ is an n co-dimensional subspace of L_p. One can verify that $L_p(A_1) \subseteq N$. Moreover, there exist k disjoint measurable subsets $B_1, B_2, \ldots, B_k$ of A_0 such that

$$N = L_p\left(T - \bigcup_{i=1}^{k} B_i\right) \bigoplus_{i=1}^{k} [g\chi_{B_i}].$$

Hence, $M = \oplus_{i=1}^{k} M_i$, where

$$M_i = \left\{ y \in L_p : \operatorname{supp}(y) \in B_i \text{ and} \int y \operatorname{sgn} g \mid g|^{p-1} \, d\mu = 0 \right\}.$$

(ii) $\Rightarrow$ (i). It follows from the following lemma and the fact that if M is a proximinal hyperplane, then P_M has a linear selection.

LEMMA 4. *Suppose M_i is a proximinal subspace of X_i and for each i, P_{M_i} admits a linear selection s_i. Then $M = (\oplus M_i)_p$ (resp. $M = (\oplus M_i)_0$), $1 \leqslant p < \infty$, (resp. $p = \infty$) is a proximinal subspace of $X = (\oplus X_i)_p$ (resp. $X = (\oplus X_i)_0$). Moreover, P_M has a linear selection $\oplus s_i$.*

Proof. For each $x_i \in X_i$, $\|s_i(x_i)\| \leqslant 2\|x_i\|$. Hence, if $(x_i) \in X$, then $(s_i(x_i)) \in M$, and $(s_i(x_i))$ is a best approximation to (x_i) from M. So P_M admits a linear selection. ∎

COROLLARY 5. *Let M be an n-co-dimensional subspace of L_p, $1 < p < \infty$. The following properties are equivalent.*

(i) *P_M admits a linear selection s.*

(ii) *There exist n disjoint measurable sets $T_1, T_2, \ldots, T_n$ such that $M = \oplus_{i=1}^{n} M_i \oplus L_p(T - \bigcup_{i=1}^{n} T_i)$, where M_i is a hyperplane of $L_p(T_i)$.*

Proof. (i) $\Rightarrow$ (ii). The range of $I - s$ is an n-dimensional subspace of L_p. Hence, (T, Σ_0, ν) is purely atomic, and there exist n disjoint sets $T_1, T_2, \ldots, T_n \in \Sigma_0$ such that $g \mid T_i \neq 0$ for each i. Therefore, $M = \oplus_{i=1}^{n} M_i \oplus L_p(T - \bigcup_{i=1}^{n} T_i)$, where

$$M_i = \left\{ y \in L_p : \operatorname{supp} y \subseteq T_i \text{ and} \int y \operatorname{sgn} g \mid g|^{p-1} \, d\mu = 0 \right\}.$$

(ii) $\Rightarrow$ (i) L_p is uniformly convex. Hence, every subspace is proximinal. By Lemma 4, P_M admits a linear selection.

3. Linear Selection in L_1

In this section, we give a characterization of those finite-dimensional subspaces of L_1 whose metric projections admit linear selections. We will need to use the following characterization of best approximations. It was first proved in the case $L_1[0, 1]$ by James [6] and in the generality stated here by Kripke and Rivlin [7].

LEMMA 6. *Let $x \in L_1 - \{0\}$. Then $0 \in P_{[y]}(x)$ if and only if*

$$\left| \int_T y \operatorname{sgn} x \, d\mu \right| \leqslant \int_{Z(x)} |y| \, d\mu.$$

Moreover, if strict inequality holds, then $P_{[y]}(x) = \{0\}$.

The following theorem extends the result of Theorem 4.4 of [2].

THEOREM 7. *Suppose that M is an n-dimensional subspace of L_1. The following properties are equivalent.*

(i) *P_M admits a linear selection.*

(ii) *There exists a subset T_1 of T which contains exactly n atoms such that for any $m \in M$*

$$\int_{T_1} |m(t)| \, d\mu \geqslant \int_{T - T_1} |m(t)| \, d\mu.$$

Proof. (i) $\Rightarrow$ (ii). Let A_0 denote a union of atoms in (T, Σ, μ) and let $A_1 = T_1 - A_0$. Since the unit ball of M is weakly compact, there exists $\delta > 0$ such that if $\mu(B) < \delta$ then $\int_B |m(t)| \, d\mu < 1/4$ for $m \in M$ and $\|m\| = 1$. A_1 is atomless; hence, there exist disjoint measurable sets B_i's such that $A_1 = \bigcup B_i$ and $\mu(B_i) < \delta$ for all i. First, we claim that if $x \in L_1$ and $\operatorname{supp} x \subseteq B_i$ for some i, then x has exactly one best approximation 0. Clearly,

$$\begin{aligned} \left| \int_T m \operatorname{sgn} x \, d\mu \right| &\leqslant \int_{B_i} |m(t)| \, d\mu \\ &\leqslant \|m\|/4 \\ &< 3\,\|m\|/4 \\ &\leqslant \int_{T - B_i} |m(t)| \, d\mu \\ &\leqslant \int_{Z(x)} |m(t)| \, d\mu. \end{aligned}$$

By Lemma 6, $P_M(x) = \{0\}$. Now we claim that there exist $t_1 \in A_0$ and $m \in M$ such that

$$|m(t_1)\,\mu(t_1)| \geqslant \int_{T-\{t_1\}} |m(t)|\,du.$$

If it were not true, by Lemma 6 each x with $\operatorname{card}(\operatorname{supp} x) = 1$ and $\operatorname{supp}(x) \subseteq A_0$ has exactly one best approximation 0. But every $m \in M$ is of the form

$$m = \sum_{t \in A_0} m\chi_{\{t\}} + \Sigma m\chi_{B_i}.$$

Hence, P_M does not admit any linear selection unless $M = \{0\}$. Choose $m_1 \in M$ so that there exists $t_1 \in A_0$ and

$$|m_1(t_1)\,\mu(t_1)| > \int_{T-\{t_1\}} |m_1(t)|\,d\mu.$$

Let $M_1 = \{m \in M : m(t_1) = 0\}$. Repeat the above argument on M_1, and $T - \{t_1\}$. There exist $m_2 \in M_1$ and $t_2 \in A_0 - \{t_1\}$ so that

$$|m_2(t_2)\,\mu(t_2)| \geqslant \int_{T-\{t_2\}} |m_2(t)|\,d\mu.$$

Let $M_2 = \{m \in M_1 : m(t_2) = 0\}$. Replace m_1 by $m_1 - m_1(t_2)\,m_2/m_2(t_2)$ if necessary. So we can assume $m_1(t_2) = 0$. By induction, there exist $m_1, m_2, \ldots, m_n \in M$ and $\{t_1, t_2, \ldots, t_n\} = T_1 \subseteq A_0$ so that $m_i(t_j) = 0$ if $i \neq j$ and

$$|m_i(t_i)\,\mu(t_i)| \geqslant \int_{T-\{t_i\}} |m_i(t)|\,d\mu = \int_{T-T_1} |m_i(t)|\,d\mu.$$

Clearly, $m_1, m_2, \ldots, m_n$ form a basis of M. And for any $m \in M$

$$\int_{T_1} |m(t)|\,d\mu \geqslant \int_{T-T_1} |m(t)|\,d\mu.$$

(ii) $\Rightarrow$ (i). Let $N = L_1(T - T_1)$. For $x \in N$ and $m \in M$

$$\int_{Z(x)} |m(t)|\,d\mu \geqslant \int_{T_1} |m(t)|\,d\mu$$

$$\geqslant \int_{T-T_1} |m(t)|\,d\mu$$

$$\geqslant \left| \int_T m(t)\,\operatorname{sgn} x(t)\,d\mu \right|.$$

By Lemma 6, $N \subseteq M^0$. Clearly, N is a complement of M. Therefore, P_M admits a linear selection. ∎

The next theorem gives a characterization of subspaces M of l_1 whose metric projections admit linear selections.

THEOREM 8. *Let M be a subspace of l_1 and let (e_i) be the natural basis of l_1. The following properties are equivalent.*

(i) *P_M admits a linear selection s.*

(ii) *There exists a subset S of $\mathbb{N}$ such that* $\operatorname{span}\{e_i : i \in S\}$ *is a complement of M and for every $m \in M$*

$$\sum_{i \notin S} |m(i)| \geqslant \sum_{i \in S} |m(i)|.$$

Proof. (i) $\Rightarrow$ (ii). Since $\mathbb{N}$ is purely atomic, the range of $I - s$ is spanned by a set of the form $\{x_i : i \in S\}$ where x_i's are pairwise disjoint and $i = \min(\operatorname{supp} x_i)$. Moreover, we may assume that $\|x_i\| = 1$. We claim that for $k \in \operatorname{supp} x_i$, $(I - s)\, e_k = \operatorname{sgn} x_i(k)\, x_i$. Suppose this claim were proved. Then for $j, k \in \operatorname{supp} x_i$, either $(I - s)(e_j + e_k) = 0$ or $(I - s)(e_j - e_k) = 0$. Thus, either $(e_j + e_k) \in M$ or $(e_j - e_k) \in M$. And the set $\{m \in M : \operatorname{supp} m \subseteq \operatorname{supp} x_i\}$ is a hyperplane of $l_1(\operatorname{supp} x_i)$. So $\operatorname{span}\{e_i : i \in S\}$ is a complement of M and

$$(I - s)\left(\sum_{i \in S} \alpha_i e_i\right) = \sum_{i \in S} \alpha_i \operatorname{sgn} x_i(i)\, x_i.$$

Hence,

$$\left\|\sum_{i \in S} \alpha_i e_i\right\| = \sum_{i \in S} |\alpha_i| = \left\|(I - s)\left(\sum_{i \in S} \alpha_i e_i\right)\right\|,$$

and $0 \in P_M\left(\sum_{i \in S} \alpha_i e_i\right)$. Therefore, by Lemma 6, for every $m \in M$ and the choice $\alpha_i = m(i)$ for $i \in S$, we have

$$\sum_{i \notin S} |m(i)| \geqslant \sum_{i \in S} |m(i)|.$$

It remains to prove that for $k \in \operatorname{supp} x_i$, $(I - s)\, e_k = \operatorname{sgn} x_i(k)\, x_i$. For every $k \in \mathbb{N}$, $(I - s)\, e_k$ is of the form $\sum_{j \in S} \alpha_j^k x_j$, and

$$\sum_{j \in S} |\alpha_j^k| = \sum_{j \in S} \alpha_j^k x_j\| \leqslant \|e_k\| = 1.$$

On the other hand,

$$\begin{aligned} x_i &= (I-s)\,x_i \\ &= (I-s)\left(\sum_{k\in\operatorname{supp}(x_i)} x_i(k)\,e_k\right) \\ &= \sum_{k\in\operatorname{supp}(x_i)} x_i(k)(I-s)\,e_k \\ &= \sum_{k\in\operatorname{supp}(x_i)} \sum_{j\in S} x_i(k)\,\alpha_j^k x_j. \end{aligned}$$

And so $\sum_{k\in\operatorname{supp}(x_i)} x_i(k)\,\alpha_i^k = 1$. Since $\sum_{k\in\operatorname{supp}(x_i)} |x_i(k)| = \|x_i\| = 1$, we must have $\alpha_i^k = \operatorname{sgn} x_i(k)$. Therefore, $\alpha_j^k = 0$ if $j \neq i$ and $(I-s)\,e_k = \operatorname{sgn} x_i(k)\,x_i$ if $k \in \operatorname{supp} x_i$.

(ii) $\Rightarrow$ (i). Let $N = l_1(\mathbb{N} - S)$. By Lemma 6, $N \subseteq M^\circ$. Since N is a complement of M, P_M admits a linear selection. ∎

Remark 1. If M is a subspace of codimension n in L_1 and if P_M admits a linear selection s, then there exist n pairwise disjoint functions $x_1, x_2, \ldots, x_n$ such that $\operatorname{range}(I-s) = \operatorname{span}\{x_i : i = 1, 2, \ldots, n\}$. Moreover, we can suppose $\|x_i\| = 1$. One can verify that if B is a measurable subset of $\operatorname{supp} x_i$, then $(I-s)(x_i\chi_B) = \|x_i\chi_B\|\,x_i$. Therefore, $\{m \in M : \operatorname{supp}(m) \subseteq \operatorname{supp}(x_i)\}$ is a hyperplane of $L_1(\operatorname{supp} x_i)$.

4. Linear Selection in $C_0(T)$

Let T be a locally compact Hausdorff space. $C_0(T)$ will denote the space of all continuous real functions x on T which "vanish at infinity" (i.e., $\{t \in T : |x(t)| \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$) and endowed with the uniform norm: $\|x\| = \sup\{|x(t)| : t \in T\}$.

If T' is a closed subset of T and $x \in C_0(T)$, then $x \mid T'$ denotes restriction of x to T'. If M is a closed subspace of $C_0(T)$, then $M' = M \mid T = \{m \mid T' : m \in M\}$ is a closed subspace of $C_0(T')$. $P_{M'}$ will denote the metric projection from $C_0(T')$ into $2^{M'}$. If T' is also an open subset, we can extend each $x' \in C_0(T')$ by

$$x(t) = \begin{cases} x'(t) & \text{if } t \in T' \\ 0 & \text{if } t \notin T'. \end{cases}$$

In this case, we will not distinguish between x' and x.

Lemma 9. *Let M be a subspace of $C_0(T)$ and $T' = \{t_1, t_2, \ldots, t_n\}$ a finite subset of T. Let $M' = M \mid T'$. If there exist $y_1', y_2' \in C(T')$ such that*

$y_i(t_j) \in \{-1, 0, 1\}$, $P_{M'}(y_i') = \{0\}$ for $i = 1, 2$, $j = 1, 2, \ldots, n$, and $0 \notin P_{M'}(y_1' + y_2')$ then P_M has no linear selection.

Proof. Since $0 \notin P_{M'}(y_1' + y_2')$ there exists $m \in M$ such that

$$\| y_1' + y_2' - m \mid T' \| < \| y_1' + y_2' \|.$$

Choose pairwise disjoint neighborhoods U_i of t_i such that if $t \in U_i$ then $m(t) \operatorname{sgn} m(t_i) \geqslant \frac{1}{2} |m(t_i)|$. By Uryshon's lemma, there exist functions $x_i \in C_0(T)$ ($i = 1, 2, \ldots, n$) such that $0 \leqslant x_i \leqslant 1$, $x_i(t_i) = 1$ and $x_i(t) = 0$ off U_i. Set $y_j = \sum_{i=1}^n y_j'(t_i) x_i$ for $j = 1, 2$. Clearly, $\| y_1 \| = \| y_2 \| = 1 = \| y_1' \| = \| y_2' \|$. (Note: $y_1 \neq 0$; otherwise $y_1' = 0$ and $P_{M'}(y_1' + y_2') = P_{M'}(y_2') = \{0\}$. Similarly, $y_2 \neq 0$.) Hence, if $z \in P_M(y_i)$ for $i = 1$ or 2, then $z \mid T' = 0$. It is also clear that $\| y_1 + y_2 \| = \| y_1' + y_2' \|$ and if $|(y_1 + y_2)(t_i)| = \| y_1 + y_2 \|$, then $m(t)(y_1(t) + y_2(t)) \geqslant 0$ for $t \in U_i$. Since $|(y_1 + y_2)(t_i)| \in \{0, 1, 2\}$,

$$\begin{aligned} &\| y_1 + y_2 - am \| \\ &\quad \leqslant \max \left(\left\| y_1' + y_2' - \frac{a}{2} \cdot m \mid T' \right\|, \| y_1 + y_2 \| - 1 + a \|m\|, a \|m\| \right) \\ &\quad < \| y_1 + y_2 \| \end{aligned}$$

for $0 < a < \min(1, 1/(2 \|m\|))$. Therefore, if $z \in P_M(y_1 + y_2)$, then $z \mid T' \neq 0$ and P_M does not admit any linear selection. ∎

It is easy to verify the following lemma.

LEMMA 10. *Let $T = \{t_1, t_2, \ldots, t_n\}$ and let M be an $n-1$-dimensional subspace of $C(T)$. If $\operatorname{card}(\sup m) \geqslant 2$ whenever $m \in M$ and $m \neq 0$, then M is Chebyshev. In this case, if $P_M(x) = 0$ and $\|x\| = 1$, then $|x(t_i)| = 1$ for $i = 1, 2, \ldots, n$.*

The following theorem extends Theorem 3.5 of [2].

THEOREM 11. *Let M be an n-dimensional subspace of $C_0(T)$. The following properties are equivalent.*

(i) *P_M admits a linear selection.*

(ii) *There exists a basis $\{m_1, m_2, \ldots, m_n\}$ of M such that $\operatorname{card}(\operatorname{supp} m_i) \leqslant 2$ for $i = 1, 2, \ldots, n$.*

(iii) *Let A_0 be the union of all isolated points of T, then there exist k disjoint subsets $B_1, B_2, \ldots, B_k$ of A_0 such that $M = \oplus_{i=1}^k M_i$, where M_i is either $C(B_i)$ or a hyperplane of $C(B_i)$.*

Proof. (i) ⇒ (ii). There exists a subset $T' = \{t_1, t_2, \ldots, t_n\}$ of T such that

$M \mid T'$ has dimension n. Therefore, $M \mid T' = C(T')$ and there exist n functions $m_1, m_2, ..., m_n$ in M such that $m_i(t_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Clearly, the m_i's form a basis of M. We claim that $\operatorname{card}(\operatorname{supp}(m_i)) \leqslant 2$. Suppose on the contrary that $\operatorname{card}(\operatorname{supp} m_1) \geqslant 3$. Let t_{n+1} and t_{n+2} be any two elements in $\operatorname{supp} m_1 - \{t_1\}$. Let $S_1 = \{i: m_i(t_{n+1}) \neq 0\}$, $S_2 = \{t: i \notin S_1$ and $m_i(t_{n+2}) \neq 0\}$, $T_1' = \{t_i : i \in S_1 \cup \{n+1\}\}$, and $T_2' = \{t_i : i \in S_2 \cup \{n+2\}\}$. Clearly, $M \mid T_1'$ has dimension $\operatorname{card}(S_1) = \operatorname{card}(T_1') - 1$, and if $m \in M$ then either $m \mid T_1' = 0$ or $\operatorname{card}(\sup(m)(T') \geqslant 2$. $M' = M \mid T_1'$ is Chebyshev in $C(T_1')$. Hence, there exist $y' \in C(T_1')$ such that $P_{M'}(y') = \{0\}$ and $|y'(t_1)| = 1$ for $t_i \in T_1'$. Similarly $M'' = \operatorname{span}\{m_i : i \in S_2\} \mid T_2'$ is Chebyshev in $C(T_2')$ and there exist $y'' \in C(T_2')$ such that $|y''(t_i)| = 1$ for $t_i \in T_2'$, and $P_{M''}(y'') = \{0\}$. Let y_i, $i = 1, 2$, in $C(T_1' \cup T_2')$ be defined by

$$y_i(t_j) = \begin{cases} y'(t_j) & \text{if } t_j \in T_1' \\ (-1)^i y''(t_j) & \text{if } t_j \in T_2'. \end{cases}$$

Let $T_1 = T_1' \cup T_2'$ and $L = M \mid T_1$. We claim that $P_L(y_i) = \{0\}$ for $i = 1$, or 2. Suppose $m' \in P_L(y_i)$. Then $m' = \sum_{i \in S_1 \cup S_2} \alpha_i m_i \mid T_i$ and $m' \mid T_1' = \sum_{i \in S_1} \alpha_i m_i \mid T_1'$ for some α_i's. Since $y_i' \mid T_1' = y'$, $\| y_i \mid T_1' - m' \mid T_1' \| \leqslant \| y_i \| = 1 = \| y_i \mid T_1' \|$ if and only if $m' \mid T_1' = 0$. Therefore, $\alpha_i = 0$ for $i \in S_1$ and $m' = \sum_{i \in S_2} \alpha_i m_i \mid T_1$. Since $y_i \mid T_2' = \pm y''$, $\|(y_i - m') \mid T_2'\| \leqslant 1 = \| y_i \|$ if and only if $\alpha_i = 0$ for $i \in S_2$. Hence, $m' = 0$. On the other hand,

$$(y_1 - y_2)(t_i) = \begin{cases} 0 & \text{if } t_i \in T_1' \\ -2y''(t) & \text{if } t_i \in T_2'. \end{cases}$$

Since $M \mid T_2' = L \mid T_2' = C(T_2')$, 0 does not belong to $P_L(y_1 - y_2)$. By Lemma 9, P_M does not admit a linear selection. This is a contradiction.

(ii) $\Rightarrow$ (iii). Let $\sim$ be an equivalence relation given by $i \sim j$ if $\operatorname{supp} m_i \cap \operatorname{supp} m_j \neq \varnothing$. If S is an equivalence class, then $M_S = \operatorname{span}\{m_i : i \in S\}$ is a hyperplane of $C(\bigcup_{i \in S} \operatorname{supp} m_i)$ unless $\operatorname{card}(\bigcup_{i \in S} \operatorname{supp}(m_i)) = \operatorname{card} S$. If S_1 and S_2 are distinct equivalent classes, then $\bigcup_{i \in S_1} \operatorname{supp}(m_i)$ and $\bigcup_{i \in S_2} \operatorname{supp}(m_i)$ are disjoint. Hence,

$$M = \bigoplus M_S \qquad \text{(sum over all equivalence classes } S)$$

and M_S is either $C(\bigcup_{i \in S} \operatorname{supp}(m_i))$ or a hyperplane of $C(\bigcup_{i \in S} \operatorname{supp}(m_i))$.

(iii) $\Rightarrow$ (i). Since M_i is a finite-dimensional subspace, M_i is proximinal. Hence, P_{M_i} admits a linear selection from $C(B_i)$ into M_i. Since $C_0(T) = \bigoplus_{i=1}^k C(B_i) \oplus C_0(T - \bigcup_{i=1}^k B_i)$ and $M = \bigoplus_{i=1}^k M_i \oplus \{0\}$, Lemma 4 P_M admits a linear selection. ∎

The following theorem gives a characterization of finite co-dimensional subspaces of c_0 whose metric projections have a linear selections.

THEOREM 12. *Suppose M is an n-co-dimensional subspace of c_0. The following properties are equivalent.*

(i) *P_M admits a linear selection s.*

(ii) *There exist n disjoint finite subsets $B_1, B_2, \ldots, B_n$ of $\mathbb{N}$ such that $M = \bigoplus_{i=1}^{n} M_i \oplus C_0(\mathbb{N} - \bigcup_{i=1}^{n} B_i)$ where M_i is a hyperplane of $C(B_i)$.*

Proof. (i) $\Rightarrow$ (ii). Since the dimension of $\ker s$ is n, we can find n vectors $y_1, y_2, \ldots, y_n \in \ker s$ and n points $t_1, t_2, \ldots, t_n \in \mathbb{N}$ such that $y_i(t_j) = \delta_{ij}$. Since $y_i \in c_0$, $i = 1, 2, \ldots, n$, there exists N such that if $m > N$ then $|y_i(m)| < 1/2n$ for $i = 1, 2, \ldots, n$. Hence, for $y \in \ker s$ and $m > N$, $|y(m)| < \|y\|/2$. Let $M' = M \mid \{1, 2, \ldots, n\}$. Then $P_{M'}$ admits a linear selection $s \mid C(\{1, 2, \ldots, N\})$. By Theorem 11, there exist k disjoint sets $B_1, B_2, \ldots, B_k$ such that $\bigcup_{i=1}^{k} B_k = \{1, 2, \ldots, N\}$ and $M' = \bigoplus_{i=1}^{k} M_i$, where M_i is a hyperplane of $C(B_k)$. We claim that if $x(i) = 0$ for $i \leqslant N$, then $x \in M$. If it were not true then $x = y + m$ for some $y \in \ker s$ and $m \in M$. Hence, $y(i) = -m(i)$ for $i \leqslant N$. But $0 \in P_M(y)$ and $|y(j)| \leqslant \|y\|/2$ for $j > N$. This is impossible. Therefore,

$$M = \bigoplus_{i=1}^{k} M_i \oplus C_0(\mathbb{N} - \{1, 2, \ldots, N\})$$

and $k = n$ (since M is an n-co-dimensional subspace).

(ii) $\Rightarrow$ (i). M_i is proximinal hyperplane of $C(B_i)$. By Lemma 4, P_M admits a linear selection. ∎

REFERENCES

1. T. ANDO, Contractive projections in L_p spaces, *Pacific J. Math* **17** (1966), 391–405.
2. F. DEUTSCH, Linear selections for the metric projection, *J. Funct. Anal.* **49** (1982), 269–292.
3. F. DEUTSCH, A survey of metric selections, *in* "Fixed Points and Nonexpansive Mappings" C. Sine, Ed.), Contemporary Math. Vol. 18, pp. 49–71, Amer. Math. Soc., Providence, R.I.
4. F. R. DEUTSCH AND P. H. MASERICK, Applications of the Hahn–Banach theorem in approximation theory, *SIAM Rev.* **9** (1967), 516–530.
5. R. B. HOLMES, "A Course on Optimization and Best Approximation," Lecture Notes in Mathematics, No. 257, Springer-Verlag, Berlin/New York, 1972.
6. R. C. JAMES, Orthogonality and linear functions in norm linear space, *Trans. Amer. Math. Soc.* **61** (1947), 265–292.
7. B. R. KRIPKE AND J. J. RIVLIN, Approximations in the metric of $L^1(X, \mu)$. *Trans. Amer. Math. Soc.* **115** (1965), 101–122.
8. H. E. LACEY, "The Isometric Theory of Classical Banach Spaces," Springer-Verlag, New York/Heidelberg/Berlin, 1974.
9. I. SINGER, "The Theory of Best Approximation and Functional Analysis," CBMS 13, SIAM, Philadelphia, 1974.